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AUTHOR(S):

LEHTO, OLLI

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Schwarzian derivative and quasiconformal mappings

Olli Lehto

Let f be a homeomorphism of the extended plane onto itself. In the theory of Teichmüller spaces it is of importance to measure the deviation of f from a Möbius transformation $z \to (az + b)/(cz + d)$ in two ways, apparently different but in fact essentially equivalent. In this connection new interesting problems have arisen of which many belong to classical complex analysis rather than to the theory of Teichmüller spaces.

1. Quasiconformal mappings

Assume that f is quasiconformal, i.e. a homeomorphic L^2-solution of a Beltrami equation $f_{\bar{z}} = \mu f_z$ with $\|\mu\|_\infty < 1$. The function μ, the complex dilatation of f, admits a simple geometric characterization: For almost all points, the derivative mapping of f at z maps concentric circles centered at z onto concentric ellipses with the ratio of axes equal to $(1 + |\mu(z)|)/(1 - |\mu(z)|)$. Every measurable μ with $\|\mu\|_\infty < 1$ is the complex dilatation of a quasiconformal f, and μ determines f up to a Möbius transformation. The case $\|\mu\|_\infty = 0$ occurs if and only if f is Möbius, and a small $\|\mu\|_\infty$ means that f is close to a Möbius transformation. (For the properties of quasiconformal mappings we refer to [8].)

2. Schwarzian derivative

Let A be a domain in the extended plane, conformally equivalent to a disc, and ρ its Poincaré-density, i.e. $\rho(z)|dz| = (1 - |w|^2)^{-1}|dw|$, $w = h(z)$, where h is a conformal map of A onto the unit disc. For a function f, meromorphic and locally injective in A, we introduce its Schwarzian derivative S_f. At finite points of A at which f takes a finite value, the definition is $S_f = (f''/f')' - (f''/f')^2/2$, and it is readily seen that this function can be continued analytically to ∞ (if it lies in A) and to the poles of f. Thus S_f is a holomorphic function in A, and direct computation shows that it vanishes identically if and only if f is Möbius. Conversely, every function φ holomorphic in A is the Schwarzian derivative of some f, and φ determines f up to a Möbius transformation.

The norm of S_f is defined by

$$\|S_f\|_A = \sup_{z \in A} |S_f(z)|/\rho(z)^2.$$

Direct computation gives the following invariance formula: If f and g are meromorphic functions in A and $h\colon B \to A$ is a conformal mapping, then

$$\|S_f - S_g\|_A = \|S_{f \circ h} - S_{g \circ h}\|_B. \tag{1}$$

In particular,

$$\|S_{h^{-1}}\|_A = \|S_h\|_B, \tag{2}$$

and we also conclude that the norm of S_f is invariant under Möbius transformations h. Like $\|\mu\|_\infty$, the norm $\|S_f\|$ also measures the deviation of f from a Möbius transformation: $\|S_f\| = 0$ if and only if f is Möbius.

3. Deviation from a disc

We call a domain a disc if it is the image of an ordinary disc under a Möbius transformation, i.e. if it is bounded by a circle or a straight line. Set

$$\sigma_1(A) = \|S_f\|_A,$$

where f is a conformal mapping of A onto a disc. Then σ_1 is well defined, and $\sigma_1(A)$ measures the deviation of A from a disc. An old theorem of Kraus (1933) says that a function f meromorphic and univalent in the unit disc D satisfies the sharp inequality $\|S_f\|_D \leq 6$. In conjunction with (2) this yields the estimate $\sigma_1(A) \leq 6$. If A is the exterior of an ellipse whose ratio of axes is r, $0 \leq r \leq 1$, then $\sigma_1(A) = 6(1 - r)/(1 + r)$. This example shows that for varying domains A, the range of σ_1 is the closed interval from 0 to 6. The value 0 occurs if and only if A itself is a disc.

We call A a quasidisc if A is the image of a disc under a quasiconformal mapping of the plane. If the mapping has maximal dilatation $\leq K$, $1 \leq K < \infty$, i.e. if its complex dilatation satisfies the inequality $(1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty) \leq K$, then A is said to be a K-quasidisc. If A is a K-quasidisc, then

$$\sigma_1(A) \leq 6\,\frac{K^2 - 1}{K^2 + 1}. \tag{3}$$

This follows from a sharpened version of Kraus's theorem: If f is quasiconformal in the plane with complex dilatation μ and conformal in a disc D, then the sharp inequality

$$\|S_f\|_D \leq 6\|\mu\|_\infty \tag{4}$$

holds (Kühnau [5], Lehto [6]). This estimate shows that if f is close to a Möbius transformation in the sense that $\|\mu\|_\infty$ is small, then the restriction $f|D$ has a small Schwarzian. To get (3) we have to combine (4) with the existence theorem for Beltrami equations which says that every function μ measurable in the plane and with $\|\mu\|_\infty < 1$ is the complex dilatation of a quasiconformal mapping. To my knowledge it is not known whether the estimate (3) is sharp.

4. Outer radius of univalency

The constant σ_1 is connected with univalency. We introduce the new domain constant

$$\sigma_2(A) = \sup \{\|S_f\|_A | f \text{ univalent in } A\},$$

the outer radius of univalency of A. For an f univalent in A we write $f = (f \circ h) \circ h^{-1}$, where h is a conformal mapping of a disc D onto A. Applying formula (1) in the case where g is the identity mapping and considering (2) we get

$$\|S_f\|_A = \|S_{f\circ h} - S_h\|_D \le \|S_{f\circ h}\|_D + \|S_{h^{-1}}\|_A \le 6 + \sigma_1(A).$$

Hence $\sigma_2(A) \le 6 + \sigma_1(A)$. A suitably chosen example shows that for every domain A equality holds here, i.e.

$$\sigma_2 = \sigma_1 + 6 \tag{5}$$

([7]).

Using (5) we obtain a generalization of (4): Let f be a mapping which is quasiconformal in the plane with complex dilatation μ and conformal in a domain A. Then

$$\|S_f\|_A \le (\sigma_1(A) + 6)\|\mu\|_\infty.$$

5. Inner radius of univalency

Let us define

$$\sigma_3(A) = \sup \{a \mid \|S_f\|_A \le a \Rightarrow f \text{ univalent in } A\},$$

the inner radius of univalency of A. Standard normal family considerations show that sup can be replaced by max, i.e. $\|S_f\|_A = \sigma_3(A)$ implies that f is univalent in A.

In 1949 Nehari [9] proved that for a disc $\sigma_3 \ge 2$, and immediately after that Hille [4] gave an example to show that here equality is true, i.e. for a disc, $\sigma_3 = 2$.

In 1963 Ahlfors [1] established the following theorem: To every quasidisc A there corresponds a positive constant η such that whenever $\|S_f\|_A < \eta$, then f is univalent in A and can be extended to a quasiconformal mapping of the plane, with $\|\mu\|_\infty = O(\|S_f\|A)$. In other words, if f is close to a Möbius transformation in the sense that $\|S_f\|$ is small, then f admits a quasiconformal extension to the plane with a small μ.

It follows from Ahlfors's theorem that for quasidiscs $\sigma_3(A) > 0$. In 1977 Gehring [2] proved that $\sigma_3(A) > 0$ only if A is a quasidisc. Put together, the results of Ahlfors and Gehring reveal an intrinsic role of quasiconformal mappings in the theory of univalent functions.

6. Universal Teichmüller space

In order to study the constant σ_3 more closely, we introduce the Banach space $Q(A)$ which consists of all functions φ holomorphic in the quasidisc A and with finite norm:

$$\|\varphi\|_A = \sup_{z \in A} |\varphi(z)|\rho(z)^2 < \infty.$$

Define the subsets

$$U(A) = \{\varphi = S_f \mid f \text{ univalent in } A\},$$

$$T(A) = \{S_f \in U(A) \mid f \text{ can be extended to a quasiconformal mapping of the plane}\}.$$

The set $T(A)$ is called the universal Teichmüller space of A. For varying domains A, the sets $U(A)$ are isomorphic: If $h: A \to A'$ is a conformal mapping and $S_f \in U(A)$, then by formula (1), $S_f \to S_{f \circ h^{-1}}$ is a bijective isometry of $U(A)$ onto $U(A')$. The same reasoning shows that the universal Teichmüller spaces of quasidiscs are isomorphic.

The universal Teichmüller space $T(A)$ is open in $Q(A)$. This follows easily from the theorem of Ahlfors cited above. Similarly, the fact that $\sigma_3(A) > 0$ only for quasidiscs implies that $T(A)$ contains the interior of $U(A)$. Hence, the results of Ahlfors and Gehring yield the relation

$$(6) \qquad T(A) = \text{interior of } U(A)$$

(For discs, this result is stated and proved in Gehring [2]).

The relation (6) allows interesting conclusions. An immediate corollary is that if $\|S_f\|_A < \sigma_3(A)$, then f can be extended to a quasiconformal mapping of the plane. We also obtain the following characterization for σ_3: If h is a conformal mapping of a disc D onto A, then $\sigma_3(A)$ is the shortest distance from the point $S_h \in T(D)$ to the boundary of $T(D)$ ([7]). This makes it possible to derive some estimates for σ_3.

As an example, let us consider the angular domain $A = \{z \mid 0 < \arg z < \alpha\pi\}$, $0 < \alpha \leq 1$. In this case, $h(z) = z^\alpha$ defines a conformal mapping of the upper half-plane D onto A, and we have

$$S_h(z) = \frac{1-\alpha^2}{2z^2}, \qquad \|S_h\|_D = 2(1-\alpha^2) < 2.$$

If $f(z) = \log z$, then S_g is not in $T(D)$, because $g(D)$ is not a Jordan domain. From $S_g(z) = 1/2z^2$ we thus deduce that

$$\sigma_3(A) = \|S_h - S_g\|_D = 2\alpha^2.$$

A similar reasoning shows that for quasidiscs A with $\sigma_1(A) < 2$,

$$\min \sigma_3(A) = 2 - \sigma_1(A).$$

We also have

$$\sigma_3(A) \leq \min (2, 6-\sigma_1(A)).$$

(For the details see [7]).

It is easy to prove that the set $U(A)$ is closed. For a long time it was a famous unsolved problem of Bers whether $U(A)$ agrees with the closure of $T(A)$. (I think that Bers announced this problem for the first time in public in a conference held in Erevan in 1965.) This was disproved by Gehring [3] in 1978. We are thus led to new problems in studying the boundary of $T(A)$. It is also an open problem whether $U(A)$ is connected.

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